

NAGATA'S EMBEDDING THEOREM

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ABSTRACT. In 1962–63, M. Nagata showed that an abstract variety could be embedded into a complete variety. Later, P. Deligne translated Nagata's proof into the language of schemes, but did not publish his notes. This paper, which is to appear as an appendix in a forthcoming book, gives an elaboration of Deligne's notes. It also contains some complementary results on extending divisors and vector sheaves to suitable completions.

The goal of this note is to prove that, if X is a scheme, separated and of finite type over a noetherian scheme S , then there exists a proper S -scheme \bar{X} and an open immersion $X \hookrightarrow \bar{X}$ over S with schematically dense image (Theorem 4.1). In addition, given a Cartier divisor D or a vector sheaf \mathcal{E} on X , the completion \bar{X} can be chosen so that D or \mathcal{E} extends to a Cartier divisor or vector sheaf (respectively) on \bar{X} .

The first assertion was proved by Nagata; see [N 1], [N 2]. Nagata's proof is phrased in terms of Zariski's language of algebraic geometry, though, which makes it difficult for many to read. Because of this, P. Deligne wrote some notes [D], which translate Nagata's work into the language of schemes. These notes, however, are unpublished.

This paper, which is based closely on Deligne's notes, was written to give a mostly self-contained exposition of the proof. After writing these notes, I encountered another (much more thorough) rendition of Deligne's notes by B. Conrad [C]. Conrad also notes the existence of another proof of Nagata's theorem by Lütkebohmert [L], which uses schemes but uses different methods.

Sections 1–4 of this note give Nagata's proof, following [D]. Section 5 adds some complementary results on constructing the completion so that certain sheaves or divisors extend to the completion.

Throughout this note all schemes are assumed to be noetherian and all ideal sheaves to be coherent. Projective morphisms are as defined in [EGA]. Schemes are not assumed to be separated unless it is explicitly mentioned.

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Given an open subscheme U of a scheme X , there are two notions of U being dense in X , and in this note the difference between them will be important. The weaker notion is that the underlying topological space of U is dense in the topological space of X . The wording “ U is a dense open subset of X ” or “ U is Zariski-dense in X ” will describe this weaker notion. A stronger notion is that the scheme-theoretic image of U under the open immersion $U \hookrightarrow X$ is all of X ; equivalently, U contains all associated points of X ¹. The wording “ U is an open dense subscheme of X ” or “ U is schematically dense in X ” will refer to this notion. This stronger notion of denseness is needed for Lemma 2.2e (for example), which is used in the proof of Theorem 2.4.

§1. Preliminary results on blowings-up

We start with some lemmas on blowings-up. For the definition of the blowing-up of a scheme X along an ideal sheaf in \mathcal{O}_X , see ([H], II § 7).

Lemma 1.1. *Let X be a scheme and let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be ideal sheaves in \mathcal{O}_X . Let \tilde{X} be the blowing-up of X along $\mathfrak{a}_1 + \dots + \mathfrak{a}_n$. Then the strict transforms of $V(\mathfrak{a}_1), \dots, V(\mathfrak{a}_n)$ in \tilde{X} have empty intersection.*

Proof. The case $n = 2$ is ([H], II Ex. 7.12); the general case is analogous. \square

Lemma 1.2. *Let U be an open subscheme of a scheme X and let \mathfrak{a}_0 be an ideal sheaf on U . Then there exists a (coherent) ideal sheaf \mathfrak{a} on X such that $\mathfrak{a}|_U = \mathfrak{a}_0$ and such that $V(\mathfrak{a}) = \overline{V(\mathfrak{a}_0)}$.*

Proof. This follows from ([EGA], I 9.5.10) and ([H], II Proposition 5.9). \square

Lemma 1.3. *Let X be a scheme and let \mathfrak{a} be an ideal sheaf in \mathcal{O}_X . Let $\pi: \tilde{X} \rightarrow X$ be the blowing-up of X along \mathfrak{a} . Then*

$$\pi^{-1}(V(\mathfrak{a})) = V(\pi^{-1}\mathfrak{a} \cdot \mathcal{O}_{\tilde{X}})$$

as closed subsets of \tilde{X} .

Proof. See ([EGA], II 8.1.8). \square

Lemma 1.4. *Let X be a scheme, and let \mathfrak{a} and \mathfrak{b} be ideal sheaves in \mathcal{O}_X . Let $\pi_1: X_1 \rightarrow X$ and $\pi_2: X_2 \rightarrow X$ be the blowings-up of X along \mathfrak{a} and $\mathfrak{a}\mathfrak{b}$, respectively. Then X_2 dominates X_1 (i.e., π_2 factors uniquely through π_1).*

Proof. The question is local on X , so we may assume that X is affine, say $X = \text{Spec } A$. Then we may regard \mathfrak{a} and \mathfrak{b} as ideals in A . Let a_0, \dots, a_n and b_0, \dots, b_m be systems of generators for \mathfrak{a} and \mathfrak{b} , respectively. Recall that $X_1 = \text{Proj } S$, where S is the graded ring $\bigoplus_{d \geq 0} \mathfrak{a}^d$ (where $\mathfrak{a}^0 = A$). This is covered by open affines $\text{Spec } S_{(a_i)}$ for $i = 0, \dots, n$. These are glued by identifying $D(a_j/a_i)$ in $\text{Spec } S_{(a_i)}$ with $D(a_i/a_j)$ in $\text{Spec } S_{(a_j)}$ via the canonical isomorphism

$$(S_{(a_i)})_{a_j/a_i} \cong (S_{(a_j)})_{a_i/a_j}.$$

¹This is shown by reducing to the affine case and using the primary decomposition.

The other blowing-up X_2 is similarly covered by open affines $\text{Spec } T_{(a_i b_j)}$, where $T = \bigoplus_{d \geq 0} \mathfrak{a}^d \mathfrak{b}^d$. The ring $S_{(a_i)}$ is generated over A by $a_0/a_i, \dots, a_n/a_i$; thus the association $a_\ell/a_i \mapsto a_\ell b_j/a_i b_j$ defines a ring homomorphism $\phi_{ij}: S_{(a_i)} \rightarrow T_{(a_i b_j)}$ for all i and j . These glue to give a morphism $X_2 \rightarrow X_1$. This morphism commutes with the maps to X because all ϕ_{ij} commute with the maps $A \rightarrow S_{(a_i)}$ and $T_{(a_i b_j)}$. \square

Proposition 1.5 ([R-G], Première partie, Lemme 5.1.4). *Let $\pi_1: X_1 \rightarrow X$ be the blowing-up of a scheme X along an ideal sheaf \mathfrak{a} in \mathcal{O}_X , and let $\pi_2: X_2 \rightarrow X_1$ be the blowing-up of X_1 along an ideal sheaf \mathfrak{b} in \mathcal{O}_{X_1} . Then there exists an ideal sheaf \mathfrak{c} in \mathcal{O}_X such that the blowing-up $\pi: \tilde{X} \rightarrow X$ of X along \mathfrak{c} is isomorphic to $\pi_2 \circ \pi_1: X_2 \rightarrow X$. Moreover, \mathfrak{c} can be chosen such that $V(\mathfrak{c}) = V(\mathfrak{a}) \cup \pi_1(V(\mathfrak{b}))$.*

Proof. Case I. X is affine. Say $X = \text{Spec } A$. Then \mathfrak{a} may be regarded as an ideal in A .

By ([EGA], II 4.4.3 and II 4.5.10), the line sheaf $\mathcal{O}(1)$ on X_1 (defined via the blowing-up π_1) is ample. Hence, for all sufficiently large integers n , $\mathfrak{b}(n) := \mathfrak{b} \otimes \mathcal{O}(n)$ is generated by global sections. Since $\mathfrak{b} \subseteq \mathcal{O}_{X_1}$, these global sections lie in $\Gamma(X_1, \mathcal{O}(n))$.

Let x_0, \dots, x_r be a system of generators for \mathfrak{a} . These determine a graded homomorphism $\phi: A[X_0, \dots, X_r] \rightarrow \bigoplus_{d \geq 0} \mathfrak{a}^d$, which in turn defines a closed immersion $f: X_1 \rightarrow \mathbb{P}_A^r$. Let $I = \ker \phi$; we then have an exact sequence of sheaves on \mathbb{P}_A^r :

$$0 \rightarrow \tilde{I} \rightarrow \mathcal{O}_{\mathbb{P}_A^r} \rightarrow f_* \mathcal{O}_{X_1} \rightarrow 0.$$

Tensor with $\mathcal{O}(n)$ and take the long exact sequence in cohomology. By Serre's theorem ([H], III 5.2), $H^1(\mathbb{P}_A^r, \tilde{I}(n)) = 0$ for all n sufficiently large; for these n , $\Gamma(\mathbb{P}_A^r, \mathcal{O}(n))$ maps surjectively onto $\Gamma(X_1, \mathcal{O}(n))$. Thus, for n sufficiently large, $\Gamma(X_1, \mathcal{O}(n)) = \mathfrak{a}^n$.

Now fix an n such that $\mathfrak{b}(n)$ is generated by global sections and such that $\Gamma(X_1, \mathcal{O}(n)) = \mathfrak{a}^n$. Let b_0, \dots, b_m be a set of global sections that generate $\mathfrak{b}(n)$; by what was shown earlier, these may be regarded as elements of \mathfrak{a}^n .

Let \mathfrak{c} be the ideal sheaf on X corresponding to the ideal

$$\mathfrak{a}^n(b_0, \dots, b_m) \subseteq A,$$

and let \tilde{X} be the blowing-up of X along \mathfrak{c} . We claim that \tilde{X} is isomorphic to X_2 . Indeed, let a_0, \dots, a_s be a system of generators for \mathfrak{a}^n , and let $S = \bigoplus_{d \geq 0} \mathfrak{a}^{nd}$. Then $X_1 = \text{Proj } S$; it is covered by open affines $\text{Spec } S_{(a_i)}$, $i = 0, \dots, s$. For each such i , the ideal $\mathfrak{b}_i \subseteq S_{(a_i)}$ corresponding to the ideal sheaf \mathfrak{b} equals

$$\mathfrak{b}_i = \left(\frac{b_0}{a_i}, \dots, \frac{b_m}{a_i} \right).$$

Let $T_i = \bigoplus_{d \geq 0} \mathfrak{b}_i^d$; then $\pi_2^{-1}(\text{Spec } S_{(a_i)}) = \text{Proj } T_i$; it is covered by open affines $\text{Spec}(T_i)_{(b_j/a_i)}$, $j = 0, \dots, m$. Also let $S' = \bigoplus_{d \geq 0} \mathfrak{c}^d$; then $\tilde{X} = \text{Proj } S'$ is covered by

open affines $\text{Spec } S'_{(a_i b_j)}$ for $i = 0, \dots, s$ and $j = 0, \dots, m$. The result then follows, in this case, since

$$(T_i)_{(b_j/a_i)} \cong S'_{(a_i b_j)}.$$

(Verification of this isomorphism is left as an exercise for the reader.)

By construction it is clear that $V(\mathfrak{c}) = V(\mathfrak{a}) \cup \pi_1(V(\mathfrak{b}))$.

Case II. General case. Let U_1, \dots, U_ℓ be a cover of X by open affines. Let $\mathfrak{c}_1, \dots, \mathfrak{c}_\ell$ be the ideal sheaves constructed as in Case I, using the same value of n . Since the construction in Case I commutes with localizing to the local ring at a point of $\text{Spec } A$, these ideal sheaves glue to give an ideal sheaf \mathfrak{c} on X with the desired properties. \square

Lemma 1.6. *Let X be a scheme, let U be an open dense subscheme, and let \mathfrak{a} be an ideal sheaf on X with $V(\mathfrak{a})$ disjoint from U . Let $\pi: \tilde{X} \rightarrow X$ denote the blowing-up of X along \mathfrak{a} . Then $\pi^{-1}(U)$ is schematically dense in \tilde{X} .*

Proof. This holds because, by ([H], II 7.13a), the inverse image ideal sheaf of \mathfrak{a} in \tilde{X} is a line sheaf. Therefore the exceptional divisor is Cartier, so locally it is generated by a function that is not a zero divisor. Thus the exceptional divisor does not contain any associated points. \square

Lemma 1.7. *Let X be an open subscheme of a scheme \bar{X} , let $p: \bar{Z} \rightarrow \bar{X}$ be a morphism of finite type, let $Z = p^{-1}(X)$, and let F and G be closed subsets of X and Z , respectively, with $p^{-1}(F) \cap G = \emptyset$.*

$$\begin{array}{ccc} G \subseteq Z & \hookrightarrow & \bar{Z} \\ \downarrow & & \downarrow p \\ F \subseteq X & \hookrightarrow & \bar{X} \end{array}$$

Then there exists an ideal sheaf \mathfrak{a} on \bar{X} , with $V(\mathfrak{a})$ disjoint from X , with the following property. Let \tilde{X} denote the blowing-up of \bar{X} along \mathfrak{a} , let $\tilde{p}: \tilde{Z} \rightarrow \tilde{X}$ be the morphism obtained from p by base change, let \tilde{F} denote the closure of F in \tilde{X} (where X is regarded as an open subscheme of \tilde{X}), and likewise let \tilde{G} be the closure of G in \tilde{Z} . Then

$$\tilde{p}^{-1}(\tilde{F}) \cap \tilde{G} = \emptyset.$$

Proof. We show that this holds for any ideal sheaf \mathfrak{a} satisfying:

- (i). $V(\mathfrak{a})$ is disjoint from X ;
- (ii). \mathfrak{a} contains the ideal corresponding to \bar{F} ; and
- (iii). \mathfrak{a} is contained in the ideal corresponding to the scheme-theoretic image $p(p^{-1}(\bar{F}) \cap \bar{G})$.

Such ideals exist: take \mathfrak{a} equal to the ideal in (iii), for example.

The above conditions are preserved upon passing to open subsets of \bar{X} and \bar{Z} , so we may assume that \bar{X} and \bar{Z} are affine, say $\bar{X} = \text{Spec } A$ and $\bar{Z} = \text{Spec } B$. Let $\phi: A \rightarrow B$ be the ring homomorphism corresponding to p . By abuse of notation, let \mathfrak{a} refer now to an ideal in the ring A . Also let \mathfrak{f} and \mathfrak{g} denote the ideals in A and B corresponding to the closed subsets \bar{F} and \bar{G} , respectively.

In the language of rings, the counterparts to conditions (ii) and (iii) above are

$$(1.7.1) \quad \mathfrak{a} \supseteq \mathfrak{f}$$

and

$$(1.7.2) \quad \mathfrak{a} \subseteq \phi^{-1}(\phi(\mathfrak{f})B + \mathfrak{g}).$$

Let f_0, \dots, f_r be a system of generators for \mathfrak{f} . As x varies over a system of generators for \mathfrak{a} , the sets $\text{Spec } S_{(x)}$ give an open affine cover of \tilde{X} , where $S = \bigoplus_{i \geq 0} \mathfrak{a}^i$. By (1.7.2) we have $\phi(x) = \sum_{i=0}^r b_i \phi(f_i) + g$ for some $b_0, \dots, b_r \in B$ and $g \in \mathfrak{g}$. We have $g \in \phi(\mathfrak{a})B$ since $g = \phi(x) - \sum b_i \phi(f_i)$, $\phi(x) \in \phi(\mathfrak{a})B$ (by choice of x), and $\phi(f_i) \in \phi(\mathfrak{a})$ by (1.7.1). Therefore we may write

$$1 = \frac{x}{x} = \frac{\sum b_i \phi(f_i)}{x} + \frac{g}{x}$$

in $B \otimes_A S_{(x)}$. But the first term lies in the ideal of $\tilde{p}^{-1}(\tilde{F})$, and the second in the ideal of \tilde{G} ; hence these two closed sets are disjoint, as was to be shown. \square

Note that the set $p(G)$ is not necessarily closed, so this cannot be proved by applying Lemma 1.1 to \bar{F} and $p(\bar{G})$.

§2. Quasi-dominations and extensions thereof

This section defines quasi-dominations and shows that they can be extended to a larger domain scheme after blowing up.

Definition 2.1. Let S be a noetherian scheme, and let X and Y be separated S -schemes of finite type.

- (a). A **quasi-domination** $f: X \dashrightarrow Y$ is a pair consisting of an open dense subscheme $U \subseteq X$ and a morphism $f: U \rightarrow Y$ whose graph is closed in $X \times_S Y$.
- (b). Such a quasi-domination is **proper** if its underlying morphism $f: U \rightarrow Y$ is proper.
- (c). If $f: X \dashrightarrow Y$ is a quasi-domination (resp. a proper quasi-domination), then we say that X **quasi-dominates** Y (resp. **properly quasi-dominates** Y) via f . Mention of f may be omitted if it is clear from the context.

Lemma 2.2. *Let S , X , and Y be as above.*

- (a). *Let $\psi: U \rightarrow Y$ be a quasi-domination $X \dashrightarrow Y$, and let V be an open subset of Y such that $\psi^{-1}(V)$ is schematically dense in X . Then $\psi|_{\psi^{-1}(V)}$ determines a quasi-domination $X \dashrightarrow V$.*
- (b). *If X quasi-dominates Y and if U is an open subset of X , then U quasi-dominates Y .*
- (c). *Let $f: X \rightarrow S$ denote the structural morphism of X , let W be an open subset of S , and suppose that Y is a scheme over W . If $f^{-1}(W)$ is schematically dense in X , then any quasi-domination $f^{-1}(W) \dashrightarrow Y$ is also a quasi-domination $X \dashrightarrow Y$.*
- (d). *Let $\psi: X \dashrightarrow Y$ be a quasi-domination, let \mathfrak{a} be an ideal sheaf on X such that $X \setminus V(\mathfrak{a})$ is schematically dense, and let $\pi: \tilde{X} \rightarrow X$ be the blowing-up of X along \mathfrak{a} . Then $\psi \circ \pi$ is a quasi-domination $\tilde{X} \dashrightarrow Y$.*
- (e). *If $U \subseteq X$ is schematically dense and $\psi_1: X \rightarrow Y$ and $\psi_2: X \rightarrow Y$ are morphisms such that $\psi_1|_U = \psi_2|_U$, then $\psi_1 = \psi_2$.*
- (f). *If U is an open dense subscheme of X and $f: U \rightarrow Y$ is proper, then f is a proper quasi-domination $X \dashrightarrow Y$.*

Proof. Part (a) follows from the fact that the intersection of the graph of ψ with $X \times_S V$ will be a closed subset of $X \times_S V$, and from the fact that the domain will still be schematically dense. Part (b) is similar. Part (c) follows from the fact that $f^{-1}(W) \times_W Y = X \times_S Y$. Part (d) follows from Lemma 1.6 and the fact that the graph of $\psi \circ \pi$ is the pull-back via $(\pi \times_S 1)$ of the graph of ψ . Part (e) follows from the fact that the inverse image of the diagonal by the morphism $(f, g): X \rightarrow Y \times_S Y$ is a closed subscheme of X containing U , hence is all of X . Finally, (f) follows from the fact that the graph of the inclusion $U \hookrightarrow X$ is a closed subset of $U \times_S X$, and that the map $U \times_S X \rightarrow Y \times_S X$ is proper. \square

Lemma 2.3. *Let X be a separated S -scheme of finite type and let U be an open dense subscheme of X . Let Y be a closed subscheme of \mathbb{A}_S^r for some $r \in \mathbb{N}$, and let $\phi: Y \hookrightarrow P$ be its scheme-theoretic closure in \mathbb{P}_S^r . Let $\psi: U \rightarrow Y$ be a morphism over S . Then there exists an ideal sheaf \mathfrak{a} on X , with $V(\mathfrak{a})$ disjoint from U , such that if $\pi: \tilde{X} \rightarrow X$ is the blowing-up of X along \mathfrak{a} , then ψ extends to a morphism $\tilde{\psi}: \tilde{X} \rightarrow P$:*

$$\begin{array}{ccc} U & \xrightarrow{\pi^{-1}|_U} & \tilde{X} \\ \downarrow \psi & & \downarrow \tilde{\psi} \\ Y & \xrightarrow{\phi} & P \end{array}$$

Proof. Let x_1, \dots, x_r be the standard coordinate functions on \mathbb{A}_S^r .

Case I. The scheme X is affine, say $X = \text{Spec } A$, and $U = D(u)$ for some $u \in A$ which is not a zero divisor. Then, since ψ is a morphism, $\psi^*x_1, \dots, \psi^*x_r$ lie in the

localized ring A_u ; hence there exists a large integer n such that

$$u^n \psi^* x_1, \dots, u^n \psi^* x_r \in A.$$

Let

$$\mathfrak{a} = (u^n, u^n \psi^* x_1, \dots, u^n \psi^* x_r).$$

Then the blowing-up \tilde{X} of X along \mathfrak{a} admits an extension of ψ to a morphism $\tilde{X} \rightarrow P$.

Case II. The complement $X \setminus U$ is the support of an effective Cartier divisor D . Let $\{W_1, \dots, W_m\}$ be an open affine cover of X such that $D|_{W_i}$ is principal for each i . By Case I there exists an ideal sheaf \mathfrak{a}_i on W_i for each i which works. These extend to ideal sheaves $\tilde{\mathfrak{a}}_i$ on X for each i by Lemma 1.2. Then by Lemma 1.4 it suffices to blow up the product of the $\tilde{\mathfrak{a}}_i$. This gives morphisms $W_i \rightarrow P$ which extend $\psi|_{W_i \cap U}: W_i \cap U \rightarrow P$. Since U is schematically dense in X , it is schematically dense in \tilde{X} ; hence these extensions are unique and therefore glue to give a morphism $\tilde{\psi}: \tilde{X} \rightarrow P$.

Case III. The lemma holds in general. Indeed, one can reduce to Case II by blowing up the ideal sheaf defining the closed subset $X \setminus U$. \square

Theorem 2.4 (Deligne [D]; cf. ([N 1], Theorem 3.2)). *Let X and Y be separated S -schemes of finite type, let U be an open dense subscheme of X , and let $\psi: U \rightarrow Y$ be a morphism over S . Then there exists an ideal sheaf \mathfrak{a} on X , with $V(\mathfrak{a})$ disjoint from U , such that if \tilde{X} is the blowing-up of X along \mathfrak{a} , then ψ extends to a quasi-domination $\tilde{\psi}: \tilde{X} \dashrightarrow Y$.*

Proof. Since Y is of finite type over S , there is an open cover $\{Y_1, \dots, Y_n\}$ of Y by open affines such that the affine ring of each Y_i is an algebra of finite type over the affine ring of some open $W_i \subseteq S$. For each i let $U_i = \psi^{-1}(Y_i)$, let $F_i = U \setminus U_i$, and let $\Gamma_i^0 \subseteq U_i \times_S Y_i$ denote the graph of $\psi|_{U_i}$. By Lemma 1.7 we may further blow up X , not touching U , such that the closure $\bar{\Gamma}_i^0$ of Γ_i^0 in $X \times_S Y_i$ does not touch $\bar{F}_i \times_S Y_i$. Let $X_i = X \setminus \bar{F}_i$.

For each i , apply Lemma 2.3 to $U_i \subseteq X_i$ and to $\psi|_{U_i}: U_i \rightarrow Y_i$. This gives a projective completion $\phi_i: Y_i \hookrightarrow P_i$ over W_i and an ideal sheaf \mathfrak{a}_i on X_i , disjoint from U_i , such that after blowing up, we have a morphism $\psi_i: X_i \rightarrow P_i$ extending $\phi_i \circ \psi|_{U_i}$. In particular, by Lemma 2.2a and Lemma 2.2c, ψ_i determines a quasi-domination $X_i \dashrightarrow Y_i$ for each i . By Lemmas 1.2, 1.4, and 2.2d, there is a blowing-up of X along an ideal sheaf \mathfrak{a} with $V(\mathfrak{a}) \cap U = \emptyset$ such that $\psi|_{U_i}$ extends to a quasi-domination $X_i \dashrightarrow Y_i$ simultaneously for all i . For each i let Γ_i denote the graph of this quasi-domination; it is a closed subset of $X_i \times_S Y_i$. Let $\Gamma \subseteq X \times_S Y$ be the union of the Γ_i .

The projection $X \times_S Y \rightarrow X$ induces isomorphisms $\Gamma_i \simeq V_i$ for some open subsets $V_i \subseteq X_i$. For all i and j the subscheme $V_i \cap V_j \cap U$ is dense in $V_i \cap V_j$; therefore Γ_i and Γ_j coincide over $V_i \cap V_j$ by Lemma 2.2e. Thus the schemes Γ_i glue to give Γ the

structure of a scheme such that the projection $X \times_S Y \rightarrow X$ induces an isomorphism $\Gamma \simeq \bigcup V_i$. Hence Γ is the graph of a morphism $\tilde{\psi}: \bigcup V_i \rightarrow Y$.

It remains to show that Γ is a closed subset. Suppose that Γ is not closed; then there exists a point $\xi \in \overline{\Gamma}$ such that $\xi \notin \Gamma$. There exists an irreducible component X' of X and an index i such that $\xi \in \overline{\Gamma_i \cap (X' \times_S Y)}$. Let η denote the generic point of X' ; then $\xi \in \overline{\{(\eta, \psi(\eta))\}}$. Pick an index j such that $\xi \in X \times_S Y_j$. We have $Y_j \cap \overline{\{\psi(\eta)\}} \neq \emptyset$, so $\psi(\eta) \in Y_j$, and therefore $\eta \in X_j$. It follows that $\overline{\{(\eta, \psi(\eta))\}} \subseteq \overline{\Gamma_j}$ and therefore that $\xi \in \overline{\Gamma_j}$. But Γ_j is a closed subset of $X_j \times_S Y_j$ and $\xi \in X \times_S Y_j$; therefore $\xi \in \overline{\Gamma_j} \times_S Y_j$. Thus the closure of Γ_j in $X \times_S Y_j$ meets $\overline{\Gamma_j} \times_S Y_j$, a contradiction.

Thus Γ determines a quasi-domination $X \dashrightarrow Y$, as was to be shown. \square

Corollary 2.5 (Chow; ([D], Cor. 1.4)). *Let X be a separated S -scheme of finite type. Let U be an open dense subset of X , quasi-projective over S . Then there exists a diagram*

$$(2.5.1) \quad \begin{array}{ccc} & X' & \hookrightarrow \overline{X} \\ U & \nearrow & \downarrow q \\ & X & \end{array}$$

in which q is a proper morphism, isomorphic over U ; $X \hookrightarrow \overline{X}$ is an open immersion; and \overline{X} is projective over S . We may also assume that the image of U is schematically dense in \overline{X} (and therefore in X').

Proof. Let W be a projective S -scheme containing U as an open dense subscheme. By Theorem 2.4 applied to the open immersion $U \hookrightarrow X$, we may assume that W quasi-dominates X . We may then take $\overline{X} = W$. Indeed, let X' denote the domain of the quasi-domination $W \dashrightarrow X$. The graph of this quasi-domination is closed in $W \times_S X$, and $W \times_S X$ is proper over X , so this graph, and hence X' , is proper over X .

Projectivity of \overline{X} follows from ([EGA], II 5.5.5, ii). \square

The special case in which X is proper over S is important enough to state separately. See also ([R-G], Première partie, Cor. 5.7.14).

Corollary 2.6. *Let X be a proper scheme over S , and let U be an open dense subset of X . If U is quasi-projective over S , then there is a proper morphism $q: X' \rightarrow X$, isomorphic over U , such that X' is projective over S and such that $q^{-1}(U)$ is schematically dense.*

Proof. Indeed, if X is proper over S then in (2.5.1) X' is proper over S , so the image of X' in \overline{X} is closed. Thus we may assume that $\overline{X} = X'$. \square

Lemma 2.7. *Let X_1 and X_2 be separated S -schemes of finite type, both of which contain open dense subschemes isomorphic to a scheme U . Suppose that there*

exist quasi-dominations $\phi: X_1 \dashrightarrow X_2$ and $\psi: X_2 \dashrightarrow X_1$ compatible with the isomorphisms with U . Then there exists a separated S -scheme X of finite type, and open immersions $X_1 \hookrightarrow X$ and $X_2 \hookrightarrow X$, such that U is schematically dense in X , compatible with $U \hookrightarrow X_i \hookrightarrow X$ for both i .

Proof. The graph Γ_1 of ϕ is (by definition) a closed subset of $X_1 \times_S X_2$, isomorphic (via the first projection) to the domain W_1 of ϕ , and containing the image of U in $X_1 \times_S X_2$. Since U is dense in X_1 , it follows that the set Γ_1 is the closure of the image of U in $X_1 \times_S X_2$. Likewise, the graph $\Gamma_2 \subseteq X_2 \times_S X_1$ of ψ is the closure of the image of U in $X_2 \times_S X_1$. These two sets are therefore the images of each other under the canonical isomorphism $X_1 \times_S X_2 \cong X_2 \times_S X_1$. Thus ϕ induces an isomorphism of W_1 with the domain W_2 of ψ , and $\psi = \phi^{-1}$.

Let X be the scheme obtained by glueing X_1 and X_2 along this isomorphism. Clearly the image of U is a dense subscheme of X ; it remains only to show that X is separated over S . Let Δ denote the diagonal subset of $X \times_S X$. The intersections of Δ with $X_1 \times_S X_1$ and $X_2 \times_S X_2$ are closed since X_1 and X_2 are separated over S . The intersections of Δ with $X_1 \times_S X_2$ and $X_2 \times_S X_1$ are closed since ϕ and ψ are quasi-dominations, respectively. Thus Δ is closed, so X is separated over S , as was to be shown. \square

Proposition 2.8 ([N 1], 4.2). *Let*

$$X_1 \hookleftarrow U \hookrightarrow X_2$$

be a diagram of separated S -schemes of finite type, and assume that U is an open dense subscheme of both X_i . Then there exists a separated S -scheme X of finite type, and an open immersion $U \hookrightarrow X$ with schematically dense image, such that X properly quasi-dominates both X_i compatible with the injections of U .

Proof. By Theorem 2.4 there exists an ideal sheaf \mathfrak{a}_1 on X_1 , with $V(\mathfrak{a}_1)$ disjoint from U , such that if $\pi_1: X'_1 \rightarrow X_1$ denotes the blowing-up along \mathfrak{a}_1 , then X'_1 quasi-dominates X_2 . Likewise, there is an ideal sheaf \mathfrak{a}_2 on X_2 such that the blowing-up $\pi_2: X'_2 \rightarrow X_2$ has the property that X'_2 quasi-dominates X_1 .

We can go up another level as follows. Let $p_1: W_1 \rightarrow X_2$ be the morphism underlying the quasi-domination $X'_1 \dashrightarrow X_2$, let \mathfrak{a}'_1 be some extension of $p_1^{-1}\mathfrak{a}_2 \cdot \mathcal{O}_{W_1}$ to X'_1 , let $\pi'_1: X''_1 \rightarrow X'_1$ denote the blowing-up of X'_1 along \mathfrak{a}'_1 , and let $W'_1 = (\pi'_1)^{-1}(W_1)$. Then, by ([H], II Cor. 7.15), p_1 extends uniquely to a morphism $p'_1: W'_1 \rightarrow X'_2$. We claim that this morphism determines a quasi-domination $X''_1 \dashrightarrow X'_2$; i.e., the graph Γ'_1 of p'_1 is closed in $X''_1 \times_S X'_2$. This graph is contained in the closed subset $(\pi'_1 \times_S \pi_2)^{-1}(\Gamma_1)$, where Γ_1 denotes the graph of p_1 . This proves the claim, since

$$\overline{\Gamma'_1} \subseteq (\pi'_1 \times_S \pi_2)^{-1}(\Gamma_1) \subseteq W'_1 \times_S X'_1,$$

and since Γ'_1 is a closed subset of $W'_1 \times_S X'_1$.

Let \mathfrak{a}'_2 and $\pi'_2: X''_2 \rightarrow X'_2$ be defined symmetrically; then $X''_2 \dashrightarrow X'_1$ is again a quasi-domination.

$$\begin{array}{ccc}
X''_1 & & X''_2 \\
\pi'_1 \downarrow & \swarrow \text{dotted} & \downarrow \pi'_2 \\
X'_1 & & X'_2 \\
\pi_1 \downarrow & \swarrow \text{dotted} & \downarrow \pi_2 \\
X_1 & & X_2
\end{array}$$

At this stage the process can be made to stop. First, note that the graphs of p'_1 and p_2 are the closures of the image of U in $X''_1 \times_S X'_2$ and $X'_2 \times_S X_1$, respectively; therefore the image of p'_1 equals the domain of p_2 . Hence

$$\begin{aligned}
(p'_1)^{-1} \mathfrak{a}'_2 \cdot \mathcal{O}_{X''_1} &= (p'_1)^{-1} (p_2^{-1} \mathfrak{a}_1 \cdot \mathcal{O}_{X'_2}) \cdot \mathcal{O}_{X''_1} \\
&= (p'_1)^{-1} p_2^{-1} \mathfrak{a}_1 \cdot (p'_1)^{-1} \mathcal{O}_{X'_2} \cdot \mathcal{O}_{X''_1} \\
&= (p_2 \circ p'_1)^{-1} \mathfrak{a}_1 \cdot \mathcal{O}_{X''_1} \\
&= (\pi_1 \circ \pi'_1)^{-1} \mathfrak{a}_1 \cdot \mathcal{O}_{X''_1} \\
&= (\pi'_1)^{-1} (\pi_1^{-1} \mathfrak{a}_1 \cdot \mathcal{O}_{X'}) \cdot \mathcal{O}_{X''_1}.
\end{aligned}$$

But $\pi_1^{-1} \mathfrak{a}_1 \cdot \mathcal{O}_{X'}$ is a line sheaf on X'_1 , so we may take \mathfrak{a}''_1 to equal the pull-back of that line sheaf to X''_1 . Thus the blowing-up X'''_1 is isomorphic to X''_1 . Constructing \mathfrak{a}''_2 and X'''_2 the same way then leads to the situation of Lemma 2.7. Therefore there exists a scheme X as in that lemma, with X''_1 and X''_2 (and hence U) schematically dense in X . Since $\pi_1 \circ \pi'_1$ and $\pi_2 \circ \pi'_2$ are proper, it follows that X properly quasi-dominates X_1 and X_2 (Lemma 2.2f). \square

§3. Some additional lemmas

This section gives some additional lemmas that will be needed for the main theorem in Section 4.

Lemma 3.1. *Let X be a scheme, separated and of finite type over a noetherian scheme S . Then there exists a cover $\{U_1, \dots, U_n\}$ of X by Zariski-dense open affines which are quasi-projective over S .*

Proof. Since X is quasi-compact, it suffices to show that each point $P \in X$ has an open neighborhood U_P which is affine, Zariski-dense, and quasi-projective over S .

First note that, since X is of finite type over S , it is covered by open affine subsets whose affine rings are of finite type over the affine rings of open affine subsets of S . In particular, these open subsets are quasi-projective over S . Each $P \in X$ has an open neighborhood of this type; also, each irreducible component of X contains an open subset of this type disjoint from all other irreducible components (take an open quasi-projective affine neighborhood of its generic point, and then localize away from

all other irreducible components of X). (Since X is noetherian, it has only finitely many irreducible components.)

Now fix $P \in X$. Let U_P^0 be an open affine quasi-projective neighborhood of P . Let U_P be the disjoint union of U_P^0 with open affine quasi-projective subsets of all irreducible components of X not contained in $\overline{U_P^0}$. This is an open affine quasi-projective Zariski-dense neighborhood of P . \square

Lemma 3.2. *Let*

$$\begin{array}{ccc} X' & \hookrightarrow & \overline{X}' \\ \downarrow p & & \downarrow q \\ X & \hookrightarrow & \overline{X} \end{array}$$

be a commutative diagram of schemes, such that p and q are separated and of finite type, $q^{-1}(X) = X'$, p is an isomorphism over an open Zariski-dense subset U of X , and the horizontal arrows are open immersions. Let F and G be closed subsets of X and \overline{X}' , respectively, such that $p^{-1}(F) \subseteq G$.

Then there exists an ideal \mathfrak{a} on \overline{X} , with $V(\mathfrak{a}) \subseteq \overline{F} \setminus F$, such that if $\pi: \tilde{X} \rightarrow \overline{X}$ denotes the blowing-up along \mathfrak{a} , if \tilde{X}' denotes the closure of U in $\overline{X}' \times_{\overline{X}} \tilde{X}$, with morphisms as below,

$$(3.2.1) \quad \begin{array}{ccc} \overline{X}' & \xleftarrow{\pi'} & \tilde{X}' \\ \downarrow q & & \downarrow \tilde{q} \\ \overline{X} & \xleftarrow{\pi} & \tilde{X} \end{array} ,$$

and if \tilde{F} denotes the closure of F in \tilde{X} , then $\tilde{q}^{-1}(\tilde{F}) \subseteq (\pi')^{-1}(G)$.

Proof. Apply Lemma 1.7 to the diagram

$$\begin{array}{ccc} \Gamma \subseteq (\overline{X}' \setminus G) \times X & \hookrightarrow & (\overline{X}' \setminus G) \times \overline{X} \\ \downarrow \text{pr}_2 & & \downarrow \text{pr}_2 \\ F \subseteq X & \hookrightarrow & \overline{X} \end{array}$$

with Γ equal to the graph of the morphism $p|_{\overline{X}' \setminus G}$. Let \tilde{F} denote as usual the closure of F in \tilde{X} , and let $\tilde{\Gamma}$ denote the closure of U in $(\overline{X}' \setminus G) \times \tilde{X}$. Then the lemma gives

$$((\overline{X}' \setminus G) \times \tilde{F}) \cap \tilde{\Gamma} = \emptyset.$$

Since $\tilde{\Gamma} \subseteq (\overline{X}' \setminus G) \times \tilde{X}$, this can be shortened to $(\overline{X}' \times \tilde{F}) \cap \tilde{\Gamma} = \emptyset$. Thus

$$(3.2.2) \quad \tilde{X}' \cap (\overline{X}' \times \tilde{F}) \subseteq \tilde{X}' \setminus \tilde{\Gamma}$$

(where \tilde{X}' is as in (3.2.1)).

On the other hand, we have $\tilde{\Gamma} = \tilde{X}' \cap ((\bar{X}' \setminus G) \times \tilde{X})$ (via the closed immersion $\bar{X}' \times_{\bar{X}} \tilde{X} \subseteq \bar{X}' \times \tilde{X}$); hence

$$\tilde{X}' \setminus \Gamma' = \tilde{X}' \cap (G \times \tilde{X}) = (\pi')^{-1}(G)$$

(where again π' is as in (3.2.1)). Also $\tilde{q}^{-1}(\tilde{F}) = \tilde{X}' \cap (\bar{X}' \times \tilde{F})$. Thus, by (3.2.2),

$$\tilde{q}^{-1}(\tilde{F}) = \tilde{X}' \cap (\bar{X}' \times \tilde{F}) \subseteq \tilde{X}' \setminus \tilde{\Gamma} = (\pi')^{-1}(G). \quad \square$$

Lemma 3.3. *For $i = 1, \dots, n$ let*

$$\begin{array}{ccc} & X_i & \hookrightarrow \bar{X}_i \\ U \hookrightarrow & \downarrow q_i & \\ & X & \end{array}$$

be a diagram of separated S -schemes of finite type, with U Zariski-dense in X , with q_i proper and isomorphic over U , with X_i Zariski-dense in \bar{X}_i , and with \bar{X}_i proper over S . Let X^* be the closure of U in the product of the \bar{X}_i , and let $p_i: X^* \rightarrow \bar{X}_i$ be the projection maps. Finally, let F_i be a closed subset of X_i for each i , such that $p_1^{-1}(F_1) \cap \dots \cap p_n^{-1}(F_n) = \emptyset$.

Then there exist ideal sheaves \mathfrak{a}_i on \bar{X}_i for each i , with $V(\mathfrak{a}_i)$ disjoint from X_i , such that after replacing each \bar{X}_i with its blowing-up along \mathfrak{a}_i , we obtain an analogous situation in which

$$p_1^{-1}(\bar{F}_1) \cap \dots \cap p_n^{-1}(\bar{F}_n) = \emptyset.$$

Proof. Let $\rho: X^{**} \rightarrow X^*$ be the blowing-up of X^* along the sum of the ideals associated to the sets $\overline{p_i^{-1}(F_i)}$. By Lemma 1.1, $\bigcap_{i=1}^n \rho^{-1}(\overline{p_i^{-1}(F_i)}) = \emptyset$.

Since the rational map $q_i \circ p_i: X^* \rightarrow X$ does not depend on i , and since q_i is proper, the set $p_i^{-1}(X_i)$ does not depend on i , and the closed set $\bigcap_{i=1}^n \overline{p_i^{-1}(F_i)}$ does not meet this open set. Thus ρ is an isomorphism over this open subset of X^* . Therefore we may identify $p_i^{-1}(X_i)$ with an open subset of X^{**} .

Now apply Lemma 3.2 to the diagrams

$$\begin{array}{ccc} p_i^{-1}(X_i) & \hookrightarrow & X^{**} \\ \downarrow & & \downarrow p_i \\ X_i & \hookrightarrow & \bar{X}_i \end{array},$$

with $F = F_i$ and $G = \overline{\rho^{-1}(p_i^{-1}(F_i))}$. This provides us with the desired blowings-up. Indeed, let $\tilde{X}_1, \dots, \tilde{X}_n$ denote the blowings-up obtained from the lemma. Let $X^\#$ be

a proper S -scheme that dominates X^{**} and the \tilde{X}_i , with corresponding morphisms $\phi: X^\# \rightarrow X^{**}$ and $\psi_i: X^\# \rightarrow \tilde{X}_i$. Finally, let \tilde{F}_i denote the closures of the F_i in \tilde{X}_i . We then have

$$\bigcap_{i=1}^n \psi_i^{-1}(\tilde{F}_i) \subseteq \bigcap_{i=1}^n \phi^{-1}(\overline{\rho^{-1}(p_i^{-1}(F_i))}) = \phi^{-1} \left(\bigcap_{i=1}^n \overline{\rho^{-1}(p_i^{-1}(F_i))} \right) = \emptyset.$$

Since the map from $X^\#$ to the new X^* is surjective, the same holds true on X^* . \square

§4. The main theorem

It is now possible to prove Nagata's embedding theorem.

Theorem 4.1. *Let X be a separated S -scheme of finite type. Then there exists an open immersion of X into a proper S -scheme, with schematically dense image.*

Proof. Let U_1, \dots, U_n be a cover of X by open Zariski-dense quasi-projective sets (Lemma 3.1). For each i , Corollary 2.5 implies that there exists a diagram

$$\begin{array}{ccc} & X_i & \hookrightarrow \bar{X}_i \\ U_i \hookrightarrow & \downarrow q_i & \\ & X & \end{array}$$

in which q_i is a proper morphism, isomorphic over U_i , and in which \bar{X}_i is proper over S . Moreover, U_i is schematically dense in \bar{X}_i .

Let $F_i = q_i^{-1}(X \setminus U_i)$, let $U = \bigcap_{i=1}^n U_i$, let X^* be the closure of U in the product of the \bar{X}_i , and let $p_i: X^* \rightarrow \bar{X}_i$ be the restrictions of the projection morphisms. By Lemma 3.3, we may assume that the \bar{X}_i are sufficiently blown up (without affecting X_i) so that

$$\bigcap_{i=1}^n p_i^{-1}(\bar{F}_i) = \emptyset.$$

For each i let M_i be the S -scheme obtained by glueing X and $\bar{X}_i \setminus \bar{F}_i$ along U_i . It is separated because Lemmas 2.2f and 2.2b imply that $\bar{X}_i \setminus \bar{F}_i$ quasi-dominates X ; hence the graph of the quasi-domination, which is just the diagonal image of U_i , is closed in $X \times_S (\bar{X}_i \setminus \bar{F}_i)$. Also X is schematically dense in M_i for all i , since U_i is schematically dense in \bar{X}_i .

Let M be the scheme obtained by applying Proposition 2.8 to the injections $X \hookrightarrow M_i$ for all i . Then X is embedded as an open dense subscheme of M . It

remains only to show that M is proper over S . Let R be a valuation ring, let K be its field of fractions, and let

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & M \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & S \end{array}$$

be a commutative diagram. Since X^* is a proper S -scheme, there is a morphism $\operatorname{Spec} R \rightarrow X^*$ making a similar diagram commute. Let i be an index such that the image does not meet $p_i^{-1}(\overline{F}_i)$. Then there exists a morphism $\operatorname{Spec} R \rightarrow M_i$ making a similar diagram commute; since M properly quasi-dominates M_i , the same holds true for M . This shows that M is proper, by the valuative criterion of properness. \square

§5. Complements: extensions of divisors and sheaves

In number theory (at least), it is sometimes useful to be able to extend an S -scheme X to a proper scheme over S such that given data extend to similar data on the larger scheme. In many cases, this follows by well-known techniques, and works for any given completion: a sheaf of ideals extends by taking the scheme-theoretic closure of the corresponding closed subscheme; a Weil divisor extends by taking the closure; ditto for an effective Weil divisor; an arbitrary sheaf, sheaf of \mathcal{O}_X -modules, or quasi-coherent sheaf extends by taking the direct image; and a coherent sheaf extends by ([H], II Ex. 5.15).

In other cases, one needs to be able to blow up the completion. For example, to extend an effective Cartier divisor D to an effective Cartier divisor on the completion, one can extend the ideal sheaf $\mathcal{O}(-D)$ to an ideal sheaf on any given completion, blow up along the extended sheaf to make the sheaf invertible, and then convert back to a Cartier divisor.

Extending arbitrary Cartier divisors and line sheaves is a bit more work, but is still fairly easy, given the work that has already been done. Extending vector sheaves involves further arguments, but still follows the same plan as for Cartier divisors. We will not treat line sheaves separately, since they occur as a special case of vector sheaves, and since the argument is parallel to that for Cartier divisors.

We start with a lemma used in extending Cartier divisors, then give a series of lemmas leading up to a parallel result for vector sheaves, and then prove the ultimate extension result for both objects simultaneously.

Lemma 5.1. *Let X be a separated S -scheme of finite type, and let U , X_1 , and X_2 be open subschemes of X with $U \subseteq X_1 \cap X_2$, $X_1 \cup X_2 = X$, and U schematically dense in X . Let D_1 and D_2 be Cartier divisors on X_1 and X_2 , respectively, that coincide on U . Then there is a proper morphism $\pi: X' \rightarrow X$ and a Cartier divisor D' on X' , such that π is an isomorphism over U , such that $\pi^{-1}(U)$ is schematically dense in X' , and such that the restriction of D' to $\pi^{-1}(U)$ coincides with the pull-backs of D_1 and D_2 .*

Proof. Let $F = (X_1 \cap X_2) \setminus U$. After a preliminary blowing-up, not affecting U , we may assume that the closure \overline{F} of F in X is the support of an effective Cartier divisor, also called \overline{F} (see ([H], II 6.13a) and Lemma 1.3). By Lemma 1.6, U is still schematically dense in the new X .

We claim that, for sufficiently large n , $D_1 - D_2 + n\overline{F}$ is an effective divisor on $X_1 \cap X_2$. By quasi-compactness it suffices to do this locally. Let $\text{Spec } A$ be an open affine in $X_1 \cap X_2$ such that \overline{F} is represented by $f \in A$ and $D_1 - D_2$ is represented by $g \in S^{-1}A$, where S is the multiplicative system of elements of A which are not zero divisors. Since $D_1 = D_2$ away from F , we actually have $g \in A_f^*$, and the claim is then obvious.

It then follows that $\mathcal{O}(-D_1 + D_2 - n\overline{F})$ is an ideal sheaf on $X_1 \cap X_2$; call it \mathfrak{a} . Extend it (by Lemma 1.2) to an ideal sheaf on all of X , and let $\pi: X' \rightarrow X$ be the blowing-up of X along this ideal sheaf. This blowing-up does not affect $X_1 \cap X_2$, and $\pi^{-1}(U)$ is schematically dense in X' . The inverse image of the extended \mathfrak{a} is then equal to $\mathcal{O}(-G)$ for some Cartier divisor G . Therefore the Cartier divisors D_1 on X_1 and $D_2 - n\overline{F} + G$ on X_2 coincide on $X_1 \cap X_2$, so they combine to give a Cartier divisor D' on X' , as was to be shown. \square

This lemma will be applied in Proposition 5.6 and Theorem 5.7 (below); the latter is the main result about extending Cartier divisors (and vector sheaves) in the context of Nagata's embedding theorem. First, though, we prove a corresponding result for vector sheaves (in which the rank-1 case is very similar to the proof of Lemma 5.1, but the higher-rank case introduces some additional difficulties).

Lemma 5.2. *Let B be a commutative ring, let $r \in \mathbb{Z}_{>0}$, and let N be a finitely generated submodule of B^r . Assume that the image of $\bigwedge^r N \rightarrow \bigwedge^r B^r \cong B$ is a principal ideal, generated by the image of a decomposable element $n_1 \wedge \cdots \wedge n_r$, and that this generator is a nonzerodivisor. Then N is a free B -module of rank r , with basis n_1, \dots, n_r .*

Proof. Let $\mathbf{e}_1, \dots, \mathbf{e}_r$ be the standard basis of B^r , write $n_i = a_{i1}\mathbf{e}_1 + \cdots + a_{ir}\mathbf{e}_r$ for all i , and let $A = (a_{ij})$ be the resulting $r \times r$ matrix. The assumptions imply that $\det A$ is a nonzerodivisor. It then follows that n_1, \dots, n_r are linearly independent over B . Indeed, if $b_1 n_1 + \cdots + b_r n_r = 0$ in N with $b_1, \dots, b_r \in B$, and if \vec{b} is the column vector composed of b_1, \dots, b_r , then ${}^t A \vec{b} = \vec{0}$. Multiplying on the left by the adjugate matrix of ${}^t A$ then gives $(\det A) \vec{b} = \vec{0}$, so $b_i = 0$ for all i since $\det A$ is a nonzerodivisor.

To show that n_1, \dots, n_r span N , let n be an arbitrary element of N , and for $i = 1, \dots, r$ let $a_i \in B$ be the (unique) element determined by the condition that the element

$$n_1 \wedge \cdots \wedge n \wedge \cdots \wedge n_r - a_i n_1 \wedge \cdots \wedge n_r$$

lies in the kernel of the map $\bigwedge^r N \rightarrow \bigwedge^r B^r$, where in the first term the n occurs in the i^{th} position. Let $n' = n - a_1 n_1 - \cdots - a_r n_r$, and write $n' = b_1 \mathbf{e}_1 + \cdots + b_r \mathbf{e}_r$ with $b_1, \dots, b_r \in B$. Then all elements $n_1 \wedge \cdots \wedge n' \wedge \cdots \wedge n_r$ lie in the kernel of $\bigwedge^r N \rightarrow \bigwedge^r B^r$, so if one replaces any row of the matrix A with b_1, \dots, b_r , then the

determinant of the resulting matrix is zero. (Here A is the same matrix as in the preceding paragraph.) Thus $b_1 C_{i1} + \cdots + b_r C_{ir} = 0$ for all i , where C_{ij} is the (i, j) cofactor matrix of A . This implies that ${}^t(\text{adj } A)\vec{b} = \vec{0}$; multiplying on the left by ${}^t A$ gives $(\det A)\vec{b} = \vec{0}$, and therefore $\vec{b} = \vec{0}$. Thus $n' = 0$, and so n lies in the span of n_1, \dots, n_r . \square

Lemma 5.3. *Let X be a noetherian scheme, let \mathcal{E} be a vector sheaf on X of rank r , let U be an open dense subscheme of X , and let \mathcal{F} be a vector subsheaf of $\mathcal{E}|_U$ (i.e., a locally free \mathcal{O}_U -submodule of $\mathcal{E}|_U$), also of rank r . Assume that the induced map $\bigwedge^r \mathcal{F} \rightarrow \bigwedge^r \mathcal{E}|_U$ is injective. Then there exists a proper morphism $\pi: X' \rightarrow X$, inducing an isomorphism $\pi^{-1}(U) \rightarrow U$, and a vector subsheaf \mathcal{F}' of $\pi^* \mathcal{E}$ on X' of rank r , such that $\mathcal{F}'|_{\pi^{-1}(U)} = \pi^* \mathcal{F}$, and such that $\pi^{-1}(U)$ is schematically dense in X' .*

Proof. The result is trivial if $r = 0$, so we assume that $r > 0$.

Following ([EGA], I 9.5.2), we let $\mathcal{G} = \ker(\mathcal{E} \rightarrow i_*(\mathcal{E}|_U/\mathcal{F}))$, where $i: U \rightarrow X$ is the inclusion map. This is a coherent sheaf by ([H], II 5.8c), ([H], II 5.7), and the fact that it is a subsheaf of the coherent sheaf \mathcal{E} . The image of $\bigwedge^r \mathcal{G}$ in $\bigwedge^r \mathcal{E}$ is a coherent subsheaf of a line sheaf, so it defines an ideal sheaf \mathfrak{a} on X . Let $\pi: X' \rightarrow X$ be the blowing-up of X along \mathfrak{a} . Since $\mathcal{G}|_U = \mathcal{F}$ and since $\bigwedge^r \mathcal{F} \rightarrow \bigwedge^r \mathcal{E}$ is injective, it follows that $\mathfrak{a}|_U$ is a line sheaf, and therefore π is an isomorphism over U . Let \mathcal{F}' be the image of $\pi^* \mathcal{G}$ in $\pi^* \mathcal{E}$. Since $\mathcal{G}|_U = \mathcal{F}$, we have $\mathcal{F}'|_{\pi^{-1}(U)} = \pi^* \mathcal{F}$.

It remains only to show that \mathcal{F}' is a vector sheaf of rank r . Let $\text{Spec } A$ be an open affine in X over which \mathcal{E} is free, fix an isomorphism $\Gamma(\text{Spec } A, \mathcal{E}) \cong A^r$, and let $\text{Spec } B$ be an open affine in $\pi^{-1}(\text{Spec } A)$. Let M be the submodule of A^r corresponding to $\mathcal{G}|_{\text{Spec } A}$, and let N be the image of $M \otimes_A B$ in B^r , so that $\mathcal{F}'|_{\text{Spec } B} = \tilde{N}$ (as subsheaves of $\pi^* \mathcal{E}|_{\text{Spec } B} = \tilde{B}^r$). The ideal $\Gamma(\text{Spec } A, \mathfrak{a})$ in A corresponds to the image of $\bigwedge^r M \rightarrow \bigwedge^r A^r \cong A$; let \mathfrak{b} denote the image of this ideal in B . By ([H], II 7.13a), \mathfrak{b} is a line sheaf, so after localizing B further we may assume that \mathfrak{b} is a principal ideal, generated by a nonzerodivisor.

We note that the submodule \mathfrak{b}_0 of $\bigwedge^r B^r$ corresponding to \mathfrak{b} is generated by elements of the form $(m_1 \wedge \cdots \wedge m_r) \otimes_A 1$ with $m_1, \dots, m_r \in M$. On the other hand, the image of $\bigwedge^r N$ in $\bigwedge^r B^r$ is generated by elements of the form $(m_1 \otimes 1) \wedge \cdots \wedge (m_r \otimes 1)$ for $m_1, \dots, m_r \in M$, and these elements correspond to the above generators for \mathfrak{b}_0 under the isomorphism $(\bigwedge^r A^r) \otimes_A B \cong \bigwedge^r (A^r \otimes_A B)$. This shows that the image of $\bigwedge^r N \rightarrow \bigwedge^r B^r \cong B$ also equals \mathfrak{b} , and therefore is principal, generated by a nonzerodivisor.

We next claim that one such (single) generator can be written as a decomposable element $(m_1 \wedge \cdots \wedge m_r) \otimes 1$, assuming that B has been chosen properly. To show this, we note that the ideal $\mathfrak{a}^* := \Gamma(\text{Spec } A, \mathfrak{a})$ is generated by elements of A of the form $\rho(m_1 \wedge \cdots \wedge m_r)$, where $\rho: \bigwedge^r A^r \xrightarrow{\sim} A$ is the usual isomorphism and $m_1, \dots, m_r \in M$. For any given choice of m_1, \dots, m_r , let $x = \rho(m_1 \wedge \cdots \wedge m_r)$,

and let $B = (\bigoplus_{d \geq 0} (\mathfrak{a}^*)^d)_{(x)}$, with x of degree 1. Then $\text{Spec } B \subseteq \pi^{-1}(\text{Spec } A)$, and $\mathfrak{b} = \mathfrak{a}^* B$ is a principal ideal, generated by x (in degree 0). We may then apply Lemma 5.2 to conclude that \mathcal{F}' is free of rank r over $\text{Spec } B$. As m_1, \dots, m_r vary, the corresponding sets $\text{Spec } B$ cover $\pi^{-1}(\text{Spec } A)$, and thus \mathcal{F}' is locally free on X' of rank r . \square

Remark 5.4. This proof also shows that $\bigwedge^r \mathcal{F}' \rightarrow \bigwedge^r \pi^* \mathcal{E}$ is injective (for the given choice of \mathcal{F}').

Lemma 5.5. *Let X be a separated S -scheme of finite type, and let U , X_1 , and X_2 be open subschemes of X with $U \subseteq X_1 \cap X_2$, $X_1 \cup X_2 = X$, and U schematically dense in X . Let \mathcal{E}_1 and \mathcal{E}_2 be vector sheaves on X_1 and X_2 , respectively, such that $\mathcal{E}_1|_U \cong \mathcal{E}_2|_U$. Then there is a proper morphism $\pi: X' \rightarrow X$ and a vector sheaf \mathcal{E}' on X' , such that π is an isomorphism over U , such that $\pi^{-1}(U)$ is schematically dense in X' , and such that $\mathcal{E}'|_{\pi^{-1}(U)} \cong \pi^*(\mathcal{E}_1|_U)$.*

Proof. By treating each connected component of U separately, we may assume that the \mathcal{E}_i have constant rank r .

After a preliminary blowing-up, not affecting U , we may assume that $X \setminus U$ is the support of an effective Cartier divisor D . Let $\phi: \mathcal{E}_1|_U \rightarrow \mathcal{E}_2|_U$ be the given isomorphism. For any open affine $\text{Spec } A$ in $X_1 \cap X_2$ on which D is principal, given say by (f) with $f \in A$, and on which \mathcal{E}_1 and \mathcal{E}_2 are trivial, the morphism ϕ is given by an invertible $r \times r$ matrix M with entries in A_f . Since D is Cartier, f is a nonzerodivisor, the map $A \rightarrow A_f$ is injective, and therefore for all sufficiently large integers n the matrix $f^n M$ extends uniquely to a matrix with entries in A . By quasi-compactness, there is one integer n which works for all such open affines $\text{Spec } A$. After replacing \mathcal{E}_2 with $\mathcal{E}_2 \otimes \mathcal{O}(nD)$, we may assume that the isomorphism ϕ extends (uniquely) to an injective morphism $\mathcal{E}_1|_{X_1 \cap X_2} \rightarrow \mathcal{E}_2|_{X_1 \cap X_2}$ of $\mathcal{O}_{X_1 \cap X_2}$ -modules.

Now let $\text{Spec } A$, f , and M be as above. Since the entries of M^{-1} also lie in A_f , there is an integer m such that the entries of $f^m M^{-1}$ all lie in A , and therefore $\det M \mid f^{rm}$ in A . In particular, $\det M$ is a nonzerodivisor, so letting $\text{Spec } A$ vary over an open cover of $X_1 \cap X_2$, we see that $\bigwedge^r \phi|_{X_1 \cap X_2}$ is injective. By Lemma 5.3, there is then a proper morphism $\pi_0: X'_2 \rightarrow X_2$, isomorphic over $X_1 \cap X_2$, and a vector subsheaf \mathcal{E}'_1 of $\pi_0^* \mathcal{E}_2$ extending the pull-back of \mathcal{E}_1 on $X_1 \cap X_2$. We then glue the X -schemes X'_2 and X_1 over $X_1 \cap X_2$ to obtain a proper morphism $\pi: X' \rightarrow X$, and glue the vector sheaves \mathcal{E}_1 on X_1 and \mathcal{E}'_1 on X'_2 to obtain the desired vector sheaf \mathcal{E}' on X' . Since U is schematically dense in $X_1 \cap X_2$ and $\pi^{-1}(X_1 \cap X_2)$ is schematically dense in X' , it follows that $\pi^{-1}(U)$ is schematically dense in X' . \square

Proposition 5.6. *Let*

$$X_1 \hookleftarrow U \hookrightarrow X_2$$

be as in Proposition 2.8, and let D be a Cartier divisor (resp. let \mathcal{E} be a vector sheaf) on U . Assume that D (resp. \mathcal{E}) extends (separately) to Cartier divisors (resp. vector sheaves) on X_1 and on X_2 . Then one may choose X in Proposition 2.8 so that D (resp. \mathcal{E}) extends to a Cartier divisor (resp. vector sheaf) on X .

Proof. For $i = 1, 2$ let D_i be a Cartier divisor on X_i that extends D (resp. let \mathcal{E}_i be a vector sheaf on X_i that extends \mathcal{E}).

By Proposition 2.8 there exists a separated S -scheme X_0 of finite type, together with an inclusion $U \hookrightarrow X_0$, such that X_0 properly quasi-dominates X_1 and X_2 , compatible with the inclusions of U into X_0 , X_1 , and X_2 . Replace X_1 and X_2 with the domains of these quasi-dominations, and D_1 and D_2 (resp. \mathcal{E}_1 and \mathcal{E}_2) with their pull-backs, so that X_1 and X_2 are open dense subschemes of X_0 . After shrinking X_0 , we may assume that $X_0 = X_1 \cup X_2$.

We then conclude by applying Lemma 5.1 (resp. Lemma 5.5). \square

Theorem 5.7. *Let X be a separated S -scheme of finite type. Then, given any Cartier divisor D (resp. vector sheaf \mathcal{E}) on X , the immersion $X \hookrightarrow \overline{X}$ of Theorem 4.1 can be chosen so that D (resp. \mathcal{E}) extends to a Cartier divisor (resp. vector sheaf) on \overline{X} .*

Proof. This will proceed by making just a few changes to the proof of Theorem 4.1. We use the notation of that proof.

The open cover U_1, \dots, U_n may be chosen so that $D|_{U_i}$ is principal for each i , say $D|_{U_i} = (f_i)$ (resp. so that \mathcal{E} is trivial on U_i). Then D extends to M_i by requiring that $D|_{\overline{X}_i \setminus \overline{F}_i} = (f_i)$ (since U_i is schematically dense in $\overline{X}_i \setminus \overline{F}_i$) (resp. \mathcal{E} extends as a trivial vector sheaf). The rest of the proof continues as before, but with Proposition 5.6 replacing Proposition 2.8. \square

Remark 5.8. When X is regular, the ability to extend a vector sheaf \mathcal{E} has already been noted ([C-G], p. 23).

Theorem 5.9. *Let X be a separated S -scheme of finite type. Then, given any finite collections D_1, \dots, D_r of Cartier divisors and $\mathcal{E}_1, \dots, \mathcal{E}_s$ of vector sheaves on X , the immersion $X \hookrightarrow \overline{X}$ of Theorem 4.1 can be chosen so that D_1, \dots, D_r and $\mathcal{E}_1, \dots, \mathcal{E}_s$ extend to \overline{X} as Cartier divisors and vector sheaves, respectively.*

Proof. First note that if $r = s = 0$ then this is just Theorem 4.1, so we may assume that $r > 0$ or $s > 0$.

Let $i_j: X \hookrightarrow \overline{X}_j$, $j = 1, \dots, r+s$, be completions for which $D_1, \dots, D_r, \mathcal{E}_1, \dots, \mathcal{E}_s$ extend, respectively. Then we may let \overline{X} be the scheme-theoretic image of the map

$$(i_1, \dots, i_{r+s}): X \hookrightarrow \overline{X}_1 \times_S \cdots \times_S \overline{X}_{r+s}$$

and pull the extended data back via the respective projection morphisms. \square

Remark 5.10. In Theorem 5.9, X is dense in the resulting completion \overline{X} , so if any of the \mathcal{E}_i have constant rank, then so do their extensions to \overline{X} .

Finally, we note that the extension of Chow's lemma given here leads to an extension of Theorem 5.9 in the (quasi-)projective setting.

Corollary 5.10. *In Theorem 5.9, if X is quasi-projective over S , then \overline{X} can be taken to be projective over S .*

Proof. Apply Corollary 2.6 to a completion $X \hookrightarrow \tilde{X}$ as in Theorem 5.9, and pull back the Cartier divisors and vector sheaves to the resulting projective scheme \overline{X} . \square

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